Bimatrix games

Transform to zero-sum game

		Colin	
		L	R
Rose	U	(-3,3)	(1,1)
	D	(5,-1)	(-1,2)

Transform to zero-sum game

$$A = \begin{pmatrix} -3 & 1 \\ 5 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\frac{1}{2}A - \frac{3}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & -2 \end{pmatrix} = -B$$

This game can be transformed to a zero-sum game

Transform to zero-sum game

If there exists α,β with α>0 such that

$$\alpha A+\beta E=-B, \quad where \quad E=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

then the game can be transformed to a zero-sum game.

Transform to zero-sum game

We may solve the game with game matrix A as solving zero-sum game.

$$A = \begin{pmatrix} -3 & 1 \\ 5 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

Nash equilibrium:

Rose plays (0.6,0.4); Payoff = 0.2

Colin plays (0.2,0.8); Payoff = 1.4

Example

		Colin	
		L	R
Rose	U	(2,-5)	(3,-7)
	D	(1,-1)	(6,4)

Example

$$\alpha A+\beta E=-B \Rightarrow \alpha\begin{pmatrix}2 & 3\\ 1 & 6\end{pmatrix}+\beta\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=-\begin{pmatrix}-5 & -7\\ -1 & 4\end{pmatrix}$$

$$\Rightarrow \begin{cases}2\alpha+\beta=5\\ \alpha+\beta=1\end{cases} \Rightarrow (\alpha,\beta)=(4,-3)$$

But $4A-3E=\begin{pmatrix}5 & 9\\ 1 & 21\end{pmatrix}\neq -B$

Therefore the game cannot be transformed to a zero-sum game.

Example

$$A=\begin{pmatrix}-1 & -7 & 5 & 3\\ 9 & -3 & 1 & -5\end{pmatrix},\ B=\begin{pmatrix}2 & 5 & -1 & 0\\ -3 & 3 & 1 & 4\end{pmatrix}$$

$$\alpha A+\beta E=-B\Rightarrow\begin{cases}-\alpha+\beta=-2\\ 9\alpha+\beta=3\end{cases}\Rightarrow(\alpha,\beta)=\left(\frac{1}{2},-\frac{3}{2}\right)$$

and $\frac{1}{2}A-\frac{3}{2}E=-B$

Therefore the game can be transformed to a zero-sum game.

Maximax

$$\begin{pmatrix} (1,5) & (8,7) & (2,9) & (9,4) \\ (7,1) & (9,6) & (8,4) & (1,2) \\ (1,9) & (2,5) & (5,8) & (8,6) \\ (8,4) & (4,9) & (6,8) & (2,8) \end{pmatrix}$$

Maximax

$$
\begin{pmatrix}
(1,5) & (6,7) & (2,8)\} & \{(9,4) \\
(7,1) & \boxed{\{(7,6)\}} & (4,4) & (1,2) \\
(1,9)\} & (2,5) & (5,8) & (8,6) \\
\{(8,4) & (4,8) & \boxed{\{(6,9)\}} & (2,8)
\end{pmatrix}
$$

Dating game

Dating Game:

$$\begin{pmatrix} (5,5) & (-1,-1) \\ (0,0) & (5,2) \end{pmatrix}$$

Dating game

Dating Game:

$$\begin{pmatrix} \{(5,5)\} & (-1,-1) \\ (0,0) & \{(5,2)\} \end{pmatrix}$$

The game also has non-pure Nash equilibrium.

Pareto optimal

Dating Game:

$$\begin{pmatrix} \{(5,5)\} & (-1,-1) \\ (0,0) & \{(5,2)\} \end{pmatrix}$$

This equilibrium point is Pareto Optimal.

Pareto optimal

An outcome of a game is non-Pareto optimal if there is another outcome which would give no player smaller payoff and give at least one of the players larger payoff. An outcome is Pareto optimal if there is no such other outcome.

Pareto optimal

		Colin	
		L	H
Rose	L	(2,2)	(5,0)
	H	(0,5)	(4,4)

The Nash equilibrium of the 'Price War' is non-Pareto Optimal.

Pareto optimal

Dating Game:

$$\begin{pmatrix} \{(5,5)\} & (-1,-1) \\ (0,0) & \{(5,2)\} \end{pmatrix}$$

This Nash equilibrium is non-Pareto Optimal.

Product and difference game

Andy and Ben choose one number from “2” and “-1”. The payoffs of Andy and Ben are the product and difference of the two numbers respectively.

Product and difference game

		Ben	
		2	-1
Andy	2	(4,0)	(-2,3)
	-1	(-2,3)	(1,0)

Product and difference game

Apply oddment method to

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \quad \begin{matrix} 6 \\ -3 \end{matrix} \quad \times \quad \begin{matrix} 1/3 \\ 2/3 \end{matrix}$$

$$\begin{matrix} 6 & -3 \\ & \times \\ \frac{1}{3} & \frac{2}{3} \end{matrix}$$

Product and difference game

Payoff to I is

$$v_A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$= 0$$

Product and difference game

Apply oddment method to

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

Player	Strategy	Payoff to I
I	$p_A = (1/3, 2/3)$	$v_A = 0$
II	$q_A = (1/3, 2/3)$	$v_A = 0$

Product and difference game

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

I	II	Payoff to I
$p_A = (1/3, 2/3)$	any	$v_A = 0$
any	$q_A = (1/3, 2/3)$	$v_A = 0$

Product and difference game

Apply oddment method to

$$B = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \quad \begin{matrix} -3 \\ 3 \end{matrix} \quad \times \quad \begin{matrix} 1/2 \\ 1/2 \end{matrix}$$

$$\begin{matrix} -3 & 3 \\ \multicolumn{2}{c}{\times} \\ \frac{1}{2} & \frac{1}{2} \end{matrix}$$

Product and difference game

Payoff to II is

$$v_B = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$$

$$= 1.5$$

Product and difference game

Apply oddment method to

$$B = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

Player	**Strategy**	**Payoff to II**
I	$p_B = (1/2, 1/2)$	$v_B = 1.5$
II	$q_B = (1/2, 1/2)$	$v_B = 1.5$

Product and difference game

$$B = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

I	II	Payoff to II
$p_B = (1/2,1/2)$	any	$v_B = 1.5$
any	$q_B = (1/2,1/2)$	$v_B = 1.5$

Product and difference game

I	II	Payoff to I	Payoff to II
$p_A = (1/3,2/3)$	any	$v_A = 0$	unknown
any	$q_B = (1/2,1/2)$	unknown	$v_B = 1.5$
$p_B = (1/2,1/2)$	any	unknown	$v_B = 1.5$
any	$q_A = (1/3,2/3)$	$v_A = 0$	unknown

Which pair of strategies (p_A, q_B) or (p_B, q_A) constitutes a Nash equilibrium?

Prudential strategies

I	II	Payoff to I	Payoff to II
$p_A = (1/3,2/3)$	any	$v_A = 0$	unknown
any	$q_B = (1/2,1/2)$	unknown	$v_B = 1.5$
$p_B = (1/2,1/2)$	any	unknown	$v_B = 1.5$
any	$q_A = (1/3,2/3)$	$v_A = 0$	unknown

The Strategies p_A and q_B are called prudential strategies.

Prudential strategies

If I uses $p_A = (1/3, 2/3)$, then since

$$p_A B = \left(\frac{1}{3}, \frac{2}{3}\right)\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

The most rational choice for II would be (1,0), i.e., II has an intention to change his strategy to (1,0).

Prudential strategies

Similarly, I has an intention to change his strategy to (1,0) since

$$Aq_B{}^T = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

Therefore the prudential strategies do not constitute a Nash equilibrium.

Prudential strategies

$$\begin{pmatrix} (4,0) & (-2,3) \\ (-2,3) & (1,0) \end{pmatrix}$$

I	II	Payoff to I	Payoff to II
(1,0)	q_B = (1/2,1/2)	1	v_B= 1.5
p_A = (1/3,2/3)	q_B = (1/2,1/2)	v_A = 0	v_B = 1.5
P_A = (1/3,2/3)	(1,0)	v_A = 0	2

p_A and q_B do not constitute a Nash equilibrium. They are called prudential strategies.

Nash equilibrium

$$\begin{pmatrix} (4,0) & (-2,3) \\ (-2,3) & (1,0) \end{pmatrix}$$

I	II	Payoff to I	Payoff to II
Any	$q_A = (1/3, 2/3)$	$v_A = 0$	**may change**
$p_B = (1/2, 1/2)$	$q_A = (1/3, 2/3)$	$v_A = 0$	$v_B = 1.5$
$p_B = (1/2, 1/2)$	**Any**	**may change**	$v_B = 1.5$

p_B and q_A constitute a Nash equilibrium.

Nash equilibrium

$$\begin{pmatrix} (4,0) & (-2,3) \\ (-2,3) & (1,0) \end{pmatrix}$$

I	II	Payoff to I	Payoff to II
$p_B = (1/2,1/2)$	$q_A = (1/3,2/3)$	$v_A = 0$	$v_B = 1.5$
$p_A = (1/3,2/3)$	(1,0)	$v_A = 0$	2
(1,0)	$q_B = (1/2,1/2)$	1	$v_B = 1.5$

The Nash equilibrium is non-Pareto optimal.

Nash equilibrium

$$\begin{pmatrix} (4,0) & (-2,3) \\ (-2,3) & (1,0) \end{pmatrix}$$

There exists strategies such that the payoffs to both players are larger.

I	II	Payoff to I	Payoff to II
$p_B = (1/2,1/2)$	$q_A = (1/3,2/3)$	$v_A = 0$	$v_B = 1.5$
(2/3,1/3)	(2/5,3/5)	0.2	1.6

Example

		Colin	
		L	R
Rose	U	(1,4)	(5,1)
	D	(4,2)	(3,3)

Example

$$A = \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix} \begin{matrix} -4 \\ 1 \end{matrix} \quad \times \quad \begin{matrix} 1/5 \\ 4/5 \end{matrix}$$

$$\begin{matrix} -3 & 2 \\ & \times \\ \frac{2}{5} & \frac{3}{5} \end{matrix}$$

Prudential strategy for I ($1/5,\ 4/5$)

Nash equilibrium for II ($\frac{2}{5},\ \frac{3}{5}$)

$$B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{matrix} 3 \\ -1 \end{matrix} \quad \times \quad \begin{matrix} 1/4 \\ 3/4 \end{matrix}$$

$$\begin{matrix} 2 & -2 \\ & \times \\ \frac{1}{2} & \frac{1}{2} \end{matrix}$$

Nash equilibrium for I ($1/4,\ 3/4$)

Prudential strategy for II ($\frac{1}{2},\ \frac{1}{2}$)

Example

$$\begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$$

Rose	Colin	Payoff to Rose	Payoff to Colin
(0,1) ↑	q_B= (1/2,1/2)	3.5	v_B= 2.5
p_A = (1/5,4/5)	q_B = (1/2,1/2) ↓	v_A = 3.4	v_B = 2.5
p_A = (1/5,4/5)	(0,1)	v_A = 3.4	2.6

Example

$$\begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$$

	Nash	Prudential
Rose	$p_B = (1/4,3/4)$	$p_A = (1/5,4/5)$
Colin	$q_A = (2/5,3/5)$	$q_B = (1/2,1/2)$
Payoff to Rose	$v_A = 3.4$	$v_A = 3.4$
Payoff to Colin	$v_B = 2.5$	$v_B = 2.5$

Pure Nash equilibrium

		Colin	
		L	R
Rose	U	(2,4)	(0,7)
	D	(3,6)	(8,5)

Pure Nash equilibrium

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 8 \end{pmatrix} \begin{matrix} 2 \\ -5 \end{matrix} \times \begin{matrix} 5/7 \\ 2/7 \end{matrix} \qquad B = \begin{pmatrix} 4 & 7 \\ 6 & 5 \end{pmatrix} \begin{matrix} 3 \\ -1 \end{matrix} \times \begin{matrix} 1/4 \\ 3/4 \end{matrix}$$

$$\begin{matrix} -1 & -8 \end{matrix} \qquad\qquad \begin{matrix} -2 & 2 \\ & \times & \\ \frac{1}{2} & \frac{1}{2} \end{matrix}$$

Same sign

Pure Nash equilibrium

$$A=\begin{pmatrix}2 & 0\\ 3 & 8\end{pmatrix}\begin{matrix}2\\ -5\end{matrix}\;\times\;\begin{matrix}5/7\\ 2/7\end{matrix}\qquad B=\begin{pmatrix}4 & 7\\ 6 & 5\end{pmatrix}\begin{matrix}3\\ -1\end{matrix}\;\times\;\begin{matrix}1/4\\ 3/4\end{matrix}$$

$$\begin{matrix}-1 & -8\end{matrix}\qquad\qquad \begin{matrix}-2 & 2\\ & \times \\ \frac{1}{2} & \frac{1}{2}\end{matrix}$$

Pure Nash equilibrium

Pure Nash equilibrium

		II	
		L	**R**
I	**U**	**(2,4)**	**(0,7)}**
	D	**{(3,6)}**	**{(8,5)**

Player I has a dominant strategy.

Pure Nash equilibrium

$$\begin{pmatrix} (2,4) & (0,7) \\ (3,6) & (8,5) \end{pmatrix}$$

	I	II	Payoff to I	Payoff to II
Nash equilibrium	(0,1)	(1,0)	3	6

Prudential strategy

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 8 \end{pmatrix} \begin{matrix} 2 \\ -5 \end{matrix} \times \begin{matrix} 5/7 \\ 2/7 \end{matrix}$$

$$\begin{matrix} -1 & -8 \end{matrix}$$

$$B = \begin{pmatrix} 4 & 7 \\ 6 & 5 \end{pmatrix} \begin{matrix} 3 \\ -1 \end{matrix} \times \begin{matrix} 1/4 \\ 3/4 \end{matrix}$$

$$\begin{matrix} -2 & 2 \\ & \times \\ \frac{1}{2} & \frac{1}{2} \end{matrix}$$

Pure Nash equilibrium

Not prudential strategy

Prudential strategy

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 8 \end{pmatrix} \begin{matrix} 2 \\ -5 \end{matrix} \times \begin{matrix} 5/7 \\ 2/7 \end{matrix}$$

$$\begin{matrix} -1 & -8 \end{matrix}$$

$$B = \begin{pmatrix} 4 & 7 \\ 6 & 5 \end{pmatrix} \begin{matrix} 3 \\ -1 \end{matrix} \times \begin{matrix} 1/4 \\ 3/4 \end{matrix}$$

$$\begin{matrix} -2 & 2 \\ \times & \\ \frac{1}{2} & \frac{1}{2} \end{matrix}$$

Pure Nash equilibrium

Pure prudential strategy

Security level

The security level is the largest payoff that a player is able to guarantee himself. In other words, it is the maximin value of player's payoff matrix when it is considered as a zero-sum game. A prudential strategy is a strategy that can guarantee the payoff not less than the security level.

Prudential strategy

$$\begin{pmatrix} (2,4) & (0,7) \\ (3,6) & (8,5) \end{pmatrix}$$

	I	II
Prudential strategy	**(0,1)**	**(1/2,1/2)**
Security Level	**3**	**5.5**

In this example, the payoff to II for the pure Nash equilibrium is 6 and is larger than the security level of II which is equal to 5.5.

Prudential strategy

$$\begin{pmatrix} (2,4) & (0,7) \\ (3,6) & (8,5) \end{pmatrix}$$

	Nash equilibrium	Prudential
I	(0,1)	(0,1)
II	(1,0)	(1/3,2/3)
Payoff to I	$v_A = 3$	$v_A = 3$
Payoff to II	6	$v_B = 5.5$

Exercise 1

$$(A, B) = \begin{pmatrix} (2,1) & (3,4) \\ (5,3) & (1,2) \end{pmatrix}$$

Exercise 1

$A=\begin{pmatrix}2 & 3\\ 5 & 1\end{pmatrix}$	$\begin{matrix}-1\\ 4\end{matrix}$	$\times$	$\begin{matrix}4/5\\ 1/5\end{matrix}$	
$\begin{matrix}-3 & 2\end{matrix}$				
$\times$				
$\begin{matrix}\frac{2}{5} & \frac{3}{5}\end{matrix}$				

- Prudential strategy for I (→ $4/5$, $1/5$)
- Nash equilibrium for II (→ $\frac{2}{5}$, $\frac{3}{5}$)

$B=\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}$	$\begin{matrix}-3\\ 1\end{matrix}$	$\times$	$\begin{matrix}1/4\\ 3/4\end{matrix}$	
$\begin{matrix}-2 & 2\end{matrix}$				
$\times$				
$\begin{matrix}\frac{1}{2} & \frac{1}{2}\end{matrix}$				

- Nash equilibrium for I (→ $1/4$, $3/4$)
- Prudential strategy for II (→ $\frac{1}{2}$, $\frac{1}{2}$)

Exercise 1

$$(A,B)=\begin{pmatrix}(2,1) & (3,4)\\(5,3) & (1,2)\end{pmatrix}$$

	Nash equilibrium	**Prudential**
I	**(0.25,0.75)**	**(0.8,0.2)**
II	**(0.4,0.6)**	**(0.5,0.5)**
Payoff to I	$v_A = 2.6$	$v_A = 2.6$
Payoff to II	$v_B = 2.5$	$v_B = 2.5$

Exercise 2

$$(A, B) = \begin{pmatrix} (1,-2) & (2,1) \\ (4,2) & (0,3) \end{pmatrix}$$

Exercise 2

$$A=\begin{pmatrix}1 & 2\\ 4 & 0\end{pmatrix}\begin{matrix}-1\\ 4\end{matrix}\times\begin{matrix}4/5\\ 1/5\end{matrix}$$

$$\begin{matrix}-3 & 2\\ & \times\\ \frac{2}{5} & \frac{3}{5}\end{matrix}$$

$$B=\begin{pmatrix}-2 & 1\\ 2 & 3\end{pmatrix}\begin{matrix}-3\\ -1\end{matrix}\times\begin{matrix}5/7\\ 2/7\end{matrix}$$

$$\begin{matrix}-4 & -2\end{matrix}$$

Pure Nash equilibrium

Exercise 2

$$\left(A,B\right)=\begin{pmatrix}\left(1,-2\right) & \left(2,1\right)\\ \left(4,2\right) & \left(0,3\right)\end{pmatrix}$$

	Nash equilibrium	Prudential
I	(1,0)	(0.8,0.2)
II	(0,1)	(0,1)
Payoff to I	2	1.6
Payoff to II	1	1.4

Competitive decision making

Zeus and Athena are two companies competing in the same market. Zeus is a big leading company while Athena is a small one. Both are trying to launch a new product with two specifications (high quality and low quality), but uncertain how large the market will be.

Competitive decision making

Large market

		Athena	
		L	H
Zeus	L	(30,10)	(28,12)
	H	(16,24)	(24,16)

Small market

		Athena	
		L	H
Zeus	L	(16,8)	(8,16)
	H	(20,4)	(16,8)

Payoffs to (Zeus, Athena)

Competitive decision making

Expected Payoff
(assuming equal chance of large and small market)

		Athena	
		L	**H**
Zeus	**L**	**(23,9)**	**(18,14)**
	H	**(18,14)**	**(20,12)**

Payoffs to (Zeus, Athena)

Competitive decision making

This is a constant sum game which can be solved as a zero sum game.

		Athena	
		L	**H**
Zeus	**L**	**(23,9)**	**(18,14)**
	H	**(18,14)**	**(20,12)**

Zeus' strategy: (2/7,5/7); payoff: 19.43

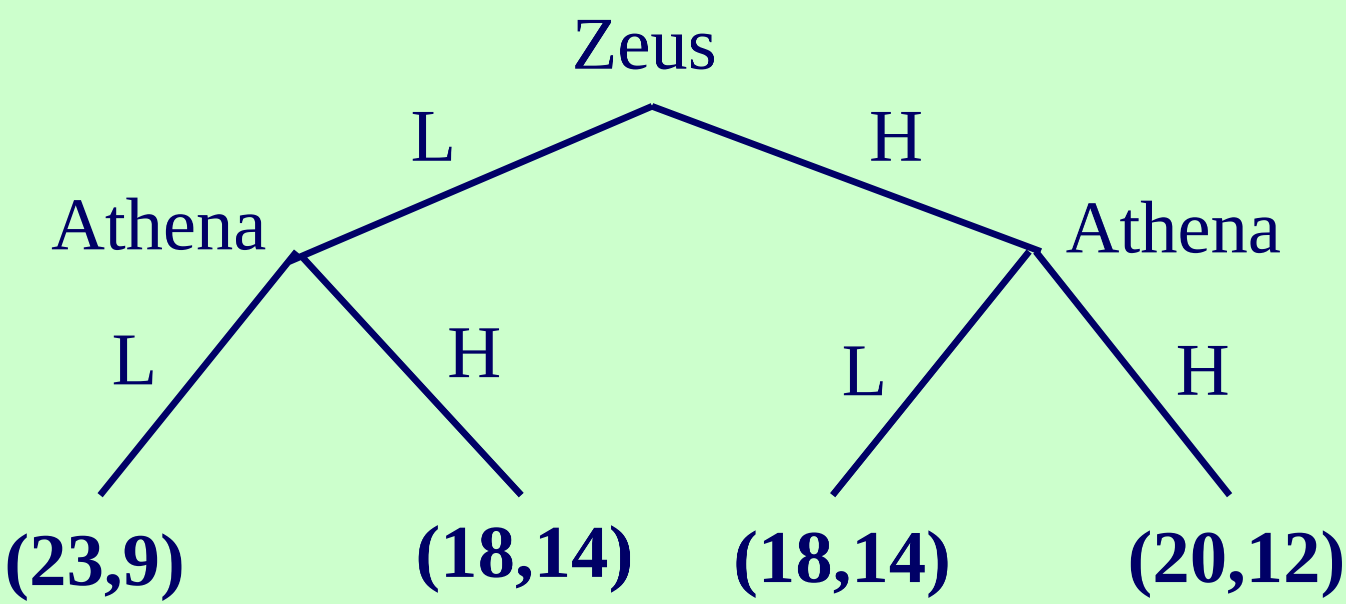
Athena's strategy: (2/7,5/7) ; payoff: 12.57

Competitive decision making

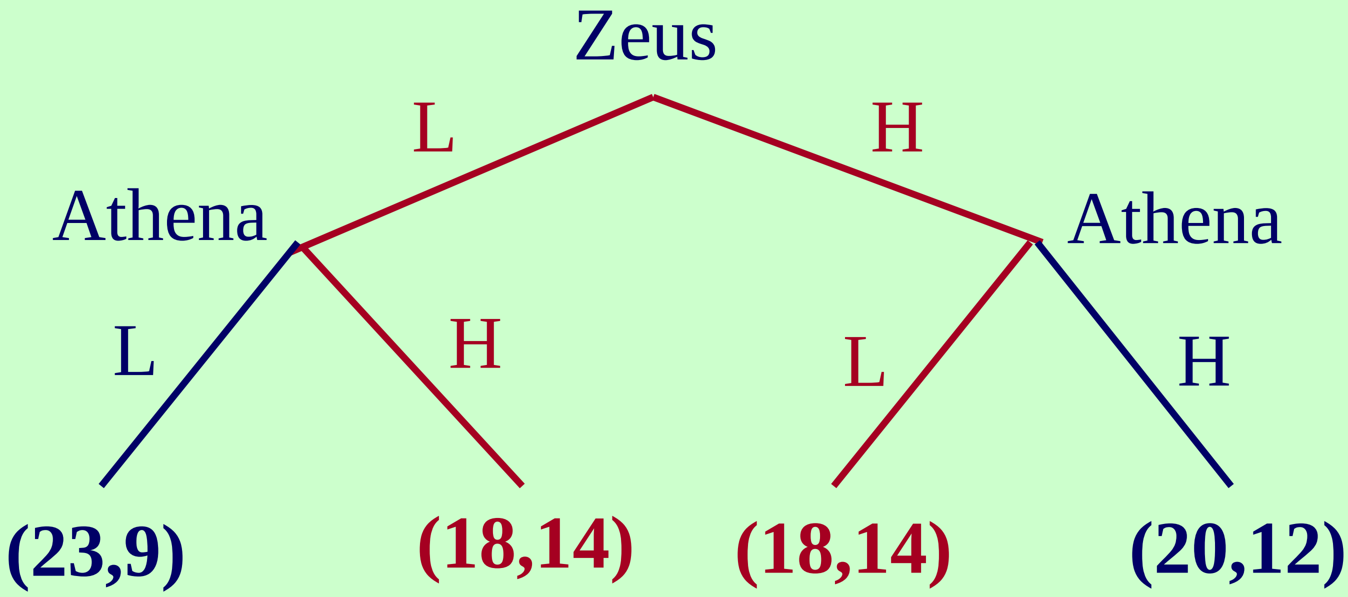
Suppose Zeus is a leading company and Athena may know Zeus's decision before it makes its own.

Competitive decision making

It becomes a sequential game



Competitive decision making



Zeus: L or H; payoff: 18

Athena: different with Zeus; payoff: 14

Making market survey

Suppose Zeus conducts a market survey to determine the market. Thus Zeus knows whether the market is large or small when it makes its decision.

Making market survey

Large market

		Athena	
		L	H
Zeus	L	(30,10)	(28,12)
	H	(16,24)	(24,16)

Small market

		Athena	
		L	H
Zeus	L	(16,8)	(8,16)
	H	(20,4)	(16,8)

Zeus: L (large) and H (small); payoff: 22
Athena: always H; payoff: 10

Making market survey

Suppose both Zeus and Athena conduct their own market surveys.

Making market survey

Large market

		Athena	
		L	H
Zeus	L	(30,10)	(28,12)
	H	(16,24)	(24,16)

Small market

		Athena	
		L	H
Zeus	L	(16,8)	(8,16)
	H	(20,4)	(16,8)

Zeus: L (large) and H (small); payoff: 22

Athena: always H; payoff: 10

Making market survey

Athena has no extra benefit by conducting her own market survey. She is able to make the right choice by knowing that Zeus has done a survey and the strategy of Zeus.

Secret survey

Suppose Zeus conduct a market survey without Athena knowing.

Secret survey

Large market

		Athena	
		L	**H**
Zeus	**L**	**(30,10)**	**(28,12)**
	H	**(16,24)**	**(24,16)**

Small market

		Athena	
		L	**H**
Zeus	**L**	**(16,8)**	**(8,16)**
	H	**(20,4)**	**(16,8)**

Zeus: L (large) and H (small); payoff: 24

Athena: different with Zeus; payoff: 8

Secret survey

It pays to know what your opponents know, but it also pays to not let your opponents know what you know.

Summary

Zeus	Athena	Zeus' strategy	Athena's strategy	Zeus' payoff	Athena's payoff
Simultaneously		(2/7,5/7)	(2/7,5/7)	19.43	12.57
First	Second	L or H	Different	18	14
Survey	No	L(l) and H(s)	H	22	10
Survey	Survey	L(l) and H(s)	H	22	10
Secret	No	L(l) and H(s)	Different	24	8